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Nonlinear Model-Following Control Application to Airplane Control

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Nonlinear model-following control design is applied to the problem of control of the six degrees of freedom of an airplane that lacks direct control of lift and side force. The nonlinear expressions for the error dynamics of the model-following control are examined using Lyapunov stability analysis. The analysis results in nonlinear feedforward and feedback gains that are functions of the airplane and model states. As a consequence, gain scheduling requirements for the implementation of the model-following control are reduced to only those involving the estimation of stability and control derivatives of the airplane. The use of these gains is shown through an example application to the control of a nonlinear aerodynamic and engine model provided by NASA Ames-Dryden Flight Research Facility. The model being followed is based on a trajectory generation algorithm, and represents a form of dynamic inversion.

Introduction

THE design methodology to be used is based on the application of nonlinear model-following to the problem of the control of the six degrees of freedom of an airplane. This methodology is related to nonlinear inverse model theory. It is a more complete approach in that it provides a means for analysis of the dynamics of the errors involved in model-following. The particular approach has been successfully applied to the control of a nonlinear aerodynamic model of a high-angle-of-attack research vehicle (HARV) through large attitude and angle of attack changes.

In general, model-following control attempts to make an actual airplane behave similarly to a prescribed mathematical model of an airplane with different force and moment characteristics than the actual airplane. The model behavior may be based on desirable flying qualities, and the matching of those flying qualities is taken to be the design objective. In this case the pilot controls are applied to the model (either conceptually or literally, to a simulation) and the airplane controls are determined.

Alternatively, the mathematical model may be a simplified representation of the actual airplane being controlled, in which case model-following control becomes a solution to the inverse problem. Here the state trajectory of the model is determined from a specification of a particular flight path or maneuver, and the airplane controls required to follow it are determined. Perfect, explicit model-following solutions to the inverse problem provide more than the open-loop controls required to fly a maneuver, since this formulation allows control of the errors between the airplane and model during the maneuver. It is this application of model-following control that is used in this paper.

To develop the nonlinear model-following controller, we will first review the model-following concepts used here. Initially a standard form of the airplane and model equations is presented with the conditions for perfect dynamic matching presented. Associated with the conditions for perfect dynamics matching are differential equations for the error. In many cases these error equations are linearized and standard linear control ideas applied to guarantee stability (i.e., they tend to go to zero in time). Hence one is led to a gain scheduling scheme. In the method presented, however, using an approach based on the stability theory of Lyapunov, a set of gains which insure stability of the nonlinear error dynamics can be found. These require no updates but are functions of the current state.

The result of this analysis is illustrated through application to the nonlinear airplane simulation provided by NASA Ames-Dryden Flight Research Facility. In this application, the model being followed is a simplified description of the airplane being controlled. The model is not, however, directly flown by externally applied (pilot) controls. Rather, it represents the states and state rates required to execute some prescribed maneuver.

Theoretical Background

Standard Form

A standard form for explicit, perfect model-following control for linear plants and models is described in Ref. 1. This standard form derives from a more general formulation applicable to nonlinear plants and models.[2] The following description of the nonlinear version of the standard form differs somewhat from the original in that it is tailored to the problem at hand, i.e., to rigid body equations of motion. Consider a plant and model whose equations of motion are separable into three types as denoted by the superscripts:

$$\begin{aligned} f_p^1(\dot{x}_p^1, x_p) &= \{0\} \\ f_p^2(\dot{x}_p^2, x_p, \eta) &= \{0\} \\ f_p^3(\dot{x}_p^3, x_p, u_p) &= \{0\} \end{aligned} \tag{1}$$

$$\begin{aligned} f_m^1(\dot{x}_m^1, x_m) &= \{0\} \\ f_m^2(\dot{x}_m^2, x_m, \eta) &= \{0\} \\ f_m^3(\dot{x}_m^3, x_m, u) &= \{0\} \end{aligned} \tag{2}$$

Type 1 equations do not contain explicit controls (such as the kinematic relations between Euler angle rates and body axis rates). Type 3 model equations are functionally dependent on the *ex deus* controls u, while type 3 plant equations are functionally dependent on the physical plant controls u_p. Type 2 plant and

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model equations are functionally dependent on some combination η of subsets of the controls u and the physical plant controls u_p.

To ensure that perfect explicit model-following is possible, we require that the type 1 and type 2 equations of the model be functionally identical to those of the plant, i.e.,

$$f_p^1(\cdot,\cdot) = f_p^1(\cdot,\cdot) \equiv f^1(\cdot,\cdot) \text{ and} \tag{3}$$

$$f_m^2(\cdot,\cdot,\cdot) = f_p^2(\cdot,\cdot,\cdot) \equiv f^2(\cdot,\cdot,\cdot) \tag{4}$$

We will also require that there be some (not necessarily unique) solution for the plant control vector, such that

$$u_p = g_p(\dot{x}_p^3, x_p) \tag{5}$$

Certain other assumptions regarding the equations must be made, all of which will be satisfied if the equations are linear in the state rate terms, as they are in the rigid body equations of motion. These assumptions may be used to show[2] that the application of the control

$$u_p = g_p(z, x_p) \tag{6}$$

to the plant equations will result in

$$\dot{x}_p^3 = z \tag{7}$$

The consequence of Eq. (7) is that the rates $\dot{x}_p^3$ are determined by specifying $z(t)$.

Further, we assume explicit solutions for the derivative terms in the type 1 and 2 equations:

$$\dot{x}_p^1 = \phi_p^1(x_p), \qquad \dot{x}_m^1 = \phi_m^1(x_m) \tag{8}$$

$$\dot{x}_p^2 = \phi_p^2(x_p, \eta), \qquad \dot{x}_m^2 = \phi_m^2(x_m, \eta) \tag{9}$$

As a consequence of Eqs. (3) and (4),

$$\phi_m^1(\cdot) = \phi_p^1(\cdot) \equiv \phi^1(\cdot) \tag{10}$$

$$\phi_m^2(\cdot,\cdot) = \phi_p^2(\cdot,\cdot) \equiv \phi^2(\cdot,\cdot) \tag{11}$$

The form of plant and model equations given by Eqs. (1) and (2), subject to the assumptions stated, is defined as the standard form for nonlinear, perfect model-following control.

The linear version of the standard form is developed in Ref. 1, where it is shown to be equivalent to transformed versions of classes of problems that satisfy the criteria for perfect model-following.[3-5]

Control Law and Error Dynamics

Given a plant and model in the standard form, it will be shown that perfect, explicit model-following results from the control

$$u_p^* = g_p(\dot{x}_{CM}^3, x_p) \tag{12}$$

In Eq. (12), the subscript CM refers to a control model which conceptually steers the plant back to the model trajectory in the presence of errors. The dynamics of the control model are related to those of the model being followed according to:

$$\begin{gathered} f^1(\dot{x}_{CM}^1, x_m) = \{0\} \qquad f^2(\dot{x}_{CM}^2, x_m, \eta) = \{0\} \\ f_m^3(\dot{x}_m^3, x_m, u) + f_{CM}(x_m, x_p) = \{0\} \end{gathered} \tag{13}$$

The difference between the control model and the original model is therefore the addition of the function $f_{CM}(x_m, x_p)$ to the type 3 model equations of motion. This function has the appearance of a full state feedback controller, wherein the feedback states are those of both the model and the plant.

By previous assumptions regarding the linearity of the model equations in the state rate variables, Eq. (13) implies that

$$\dot{x}_{CM}^1 = \dot{x}_m^1, \quad \dot{x}_{CM}^2 = \dot{x}_m^2, \quad \dot{x}_{CM}^3 = \dot{x}_m^3 + f_{CM}(x_m, x_p) \tag{14}$$

The function $f_{CM}(x_m, x_p)$ is typically (but not necessarily) a function of the error between the model and the plant trajectories, and at any rate is taken such that it vanishes whenever the error is zero. That is,

$$f_{CM}(x_m, x_m) = \{0\} \tag{15}$$

so that

$$\dot{x}_{CM}^3\big|_{e=\{0\}} = \dot{x}_m^3 \tag{16}$$

To demonstrate that application of the control law [Eq. (12)] results in perfect model-following, we examine the error dynamics of the system. Consider the error between model and plant,

$$e = \begin{Bmatrix} e^1 \\ e^2 \\ e^3 \end{Bmatrix} = \begin{Bmatrix} x_m^1 - x_p^1 \\ x_m^2 - x_p^2 \\ x_m^3 - x_p^3 \end{Bmatrix} \tag{17}$$

and the error rate,

$$\dot{e} = \begin{Bmatrix} \dot{e}^1 \\ \dot{e}^2 \\ \dot{e}^3 \end{Bmatrix} = \begin{Bmatrix} \dot{x}_m^1 - \dot{x}_p^1 \\ \dot{x}_m^2 - \dot{x}_p^2 \\ \dot{x}_m^3 - \dot{x}_p^3 \end{Bmatrix} \tag{18}$$

With respect to the control model,

$$\begin{aligned} \dot{x}_{CM}^1 - \dot{x}_p^1 &= \dot{x}_m^1 - \dot{x}_p^1 \\ &= \dot{e}^1 \\ &= \phi^1(x_m) - \phi^1(x_p) \end{aligned} \tag{19}$$

$$\begin{aligned} \dot{x}_{CM}^2 - \dot{x}_p^2 &= \dot{x}_m^2 - \dot{x}_p^2 \\ &= \dot{e}^2 \\ &= \phi^2(x_m, \eta) - \phi^2(x_p, \eta) \end{aligned} \tag{20}$$

$$\begin{aligned} \dot{x}_{CM}^3 - \dot{x}_p^3 &= \dot{x}_m^3 + f_{CM}(x_m, x_p) - \dot{x}_p^3 \\ &= \dot{e}^3 + f_{CM}(x_m, x_p) \\ &= \{0\} \end{aligned} \tag{21}$$

The results in Eqs. (19) and (20) are based on the relationships between the control model and the model in Eq. (14), and the definitions in Eqs. (10) and (11). Equation (21) results from the definition of the control model in Eq. (14), the assertions in Eqs. (6) and (7), with the substitution $z = \dot{x}_{CM}^3$ in Eq. (12).

We may therefore write the error dynamics as:

$$\dot{e} = \begin{Bmatrix} \phi^1(x_m) - \phi^1(x_p) \\ \phi^2(x_m, \eta) - \phi^2(x_p, \eta) \\ -f_{CM}(x_m, x_p) \end{Bmatrix} \tag{22}$$

As a consequence of the restrictions placed on the model, and on the construction of the control model, the right-hand side of Eq. (22) vanishes when the model and plant are aligned, or when the error is zero. The evolution of the error when it is nonzero depends on the problem at hand.

The problem is to determine, if possible, the functions $f_{CM}(x_m, x_p)$ [subject to $f_{CM}(x_m, x_m) = \{0\}$] such that the error dynamics as given by Eq. (22) satisfy design criteria. One means of accomplishing this is through selection of an appropriate Lyapunov function. Alternatively, Eq. (22) may be linearized about the condition of zero error, and the control model functions appear as linear feedback gains on the error regulator problem.

Application to Airplane Control

States and Controls

The states used are q_0, q_1, q_2, q_3, P, Q, R, U, V, W, defined as follows: $q_0 \ldots q_3$, Euler parameters, body-to-earth (inertial) transformation, P, Q, R, body axis roll, pitch, and yaw rates, and U, V, W, body axis X, Y, and Z velocities.

The control vector ($\boldsymbol{u}_p$) is comprised of all the independently acting force and moment effectors on the airplane, and has dimension m.

Equations of Motion

Euler parameters have been selected for the formulation of the equations of motion. This is done because of the simplifications in the error dynamics equations that result. The rigid body equations of motion are as given in Eqs. (23–32). With appropriate subscripts on the forces, moments, states, and state rates, the same set of equations will apply to either the airplane, the model, or the control model.

$$0 = \tfrac{1}{2}(-q_1P - q_2Q - q_3R) - \dot{q}_0 \tag{23}$$

$$0 = \tfrac{1}{2}(q_0P - q_3Q + q_2R) - \dot{q}_1 \tag{24}$$

$$0 = \tfrac{1}{2}(q_3P + q_0Q - q_1R) - \dot{q}_2 \tag{25}$$

$$0 = \tfrac{1}{2}(-q_2P + q_1Q + q_0R) - \dot{q}_3 \tag{26}$$

$$0 = I_{xx}\dot{P} - I_{xz}(\dot{R} + PQ) - (I_{yy} - I_{zz})QR - L(\boldsymbol{x}, \dot{\boldsymbol{x}}, \boldsymbol{u}) \tag{27}$$

$$0 = I_{yy}\dot{Q} - I_{xz}(R^2 - P^2) - (I_{zz} - I_{xx})PR - M(\boldsymbol{x}, \dot{\boldsymbol{x}}, \boldsymbol{u}) \tag{28}$$

$$0 = I_{zz}\dot{R} - I_{xz}(\dot{P} - QR) - (I_{xx} - I_{yy})PQ - N(\boldsymbol{x}, \dot{\boldsymbol{x}}, \boldsymbol{u}) \tag{29}$$

$$0 = -mg\mathrm{T}^{31} + m(\dot{U} + QW - RV) - X(\boldsymbol{x}, \dot{\boldsymbol{x}}, \boldsymbol{u}) \tag{30}$$

$$0 = -mg\mathrm{T}^{32} + m(\dot{V} + RU - PW) - Y(\boldsymbol{x}, \dot{\boldsymbol{x}}, \boldsymbol{u}) \tag{31}$$

$$0 = -mg\mathrm{T}^{33} + m(\dot{W} + PV - QU) - Z(\boldsymbol{x}, \dot{\boldsymbol{x}}, \boldsymbol{u}) \tag{32}$$

Equations (23–26) are the Euler parameter equivalents to the usual Euler angle rate equations. The T^{ij} in Eqs. (30–32) are the elements of the body-to-Earth transformation matrix:

$$\mathrm{T}_{EB} = \begin{bmatrix} (q_0^2 + q_1^2 - q_2^2 - q_3^2) & 2(q_1q_2 - q_0q_3) & 2(q_1q_3 + q_0q_2) \\ 2(q_1q_2 + q_0q_3) & (q_0^2 - q_1^2 + q_2^2 - q_3^2) & 2(q_2q_3 - q_0q_1) \\ 2(q_1q_3 - q_0q_2) & 2(q_2q_3 + q_0q_1) & (q_0^2 - q_1^2 - q_2^2 + q_3^2) \end{bmatrix} \tag{33}$$

The forces and moments appearing in Eqs. (27–30) are assumed to be separable into those arising from control application (superscript C), and those arising from the airframe interaction with the atmosphere (superscript A):

$$\begin{aligned} L(\boldsymbol{x}, \dot{\boldsymbol{x}}, \boldsymbol{u}_p) &= L^C(\boldsymbol{u}_p) + L^A(\boldsymbol{x}, \dot{\boldsymbol{x}}) \\ &\vdots \\ X(\boldsymbol{x}, \dot{\boldsymbol{x}}, \boldsymbol{u}_p) &= X^C(\boldsymbol{u}_p) + X^A(\boldsymbol{x}, \dot{\boldsymbol{x}}) \end{aligned} \tag{34}$$

The representation of the forces and moments in Eq. (34) is consistent with the usual Taylor series expansions that result in stability and control derivatives.

Control Law

Note that Eqs. (23–26) for the airplane and for the model are functionally identical, do not depend on the controls, and are linear in the state rate terms. Thus they are type 1 equations of the standard form.

If we lack direct control of side force and lift, we must relegate Eqs. (31) and (32) to the status of type 2 equations. The practical effect of this is to require that the model have exactly the same aerodynamic and control generated lift and side-force characteristics as the airplane.

Now, if a solution for the plant control vector in Eqs. (27–30) can be found [Eq. (5)], they will satisfy the requirements for the type 3 equations of the standard form. Finally, if a suitable control model function [Eq. (15)] that stabilizes the errors can be determined, then perfect model following is achievable.

Consistent with the partitioning of the standard form, the state vector is defined as

$$\boldsymbol{x} = \begin{Bmatrix} \boldsymbol{x}^1 \\ \boldsymbol{x}^2 \\ \boldsymbol{x}^3 \end{Bmatrix} = \begin{Bmatrix} \{q_0\ q_1\ q_2\ q_3\}^{\mathrm{T}} \\ \{V\ W\}^{\mathrm{T}} \\ \{P\ Q\ R\ U\}^{\mathrm{T}} \end{Bmatrix} \tag{35}$$

$$\boldsymbol{x}^1 \equiv \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \begin{Bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{Bmatrix} \tag{36}$$

$$\boldsymbol{x}^2 \equiv \begin{Bmatrix} x_5 \\ x_6 \end{Bmatrix} = \begin{Bmatrix} V \\ W \end{Bmatrix} \tag{37}$$

$$\boldsymbol{x}^3 \equiv \begin{Bmatrix} x_7 \\ x_8 \\ x_9 \\ x_{10} \end{Bmatrix} = \begin{Bmatrix} P \\ Q \\ R \\ U \end{Bmatrix} \tag{38}$$

Throughout the following, vectors and matrices will be partitioned in a like manner.

We now assume that the airplane's control generated forces and moments are adequately described by first-order (Taylor series) approximations in the usual manner. Equations (27–30) can then be represented as (with subscript p to indicate the airplane):

$$\left[\frac{\partial \boldsymbol{F}_p^{\mathrm{Control}}}{\partial \boldsymbol{u}_p}\right]_{\mathrm{Ref}} \boldsymbol{u}_p = \begin{Bmatrix} -I_{xz}PQ - (I_{yy} - I_{zz})QR - L^A + I_{xx}\dot{P} - I_{xz}\dot{R} \\ -I_{xz}(R^2 - P^2) - (I_{zz} - I_{xx})PR - M^A + I_{yy}\dot{Q} \\ I_{xz}QR - (I_{xx} - I_{yy})PQ - N^A + I_{zz}\dot{R} - I_{xz}\dot{P} \\ 2mg(q_0q_2 - q_1q_3) + m(QW - RV) - X^A + m\dot{U} \end{Bmatrix}_P \tag{39}$$

Here, $\boldsymbol{F}_p$ is the vector of directly controlled moments and forces, $\boldsymbol{F}_p^T = \{L_p\ M_p\ N_p\ X_p\}$. For convenience, the vectoredvalued function on the right-hand side is separated into two functions, $f_1(x)$ and $f_2(\dot{x})$:

$$\left[\frac{\partial F_p^{\text{Control}}}{\partial u_p}\right]_{\text{Ref}} u_p = f_1(x_p) + f_2(\dot{x}_p) \quad (40)$$

We now seek an expression for the control vector u_p. The matrix of partial derivatives on the left side of Eq. (40) is wide (more independent controllers than directly controlled degrees of freedom). The rank of this matrix depends on whether the controllers are independent in their actions.

Here we assume that the control matrix is of full rank (rank = 4), and assume the existence of a solution for the control vector. This may be the minimum norm solution or any other generalized inverse suitable to the problem.[6,7]

$$u_p = \left[\frac{\partial F_p^{\text{Control}}}{\partial u_p}\right]^*_{\text{Ref}} \{f_1(x_p) + f_2(\dot{x}_p)\} \quad (41)$$

where $[\partial F_p^{\text{Control}}/\partial u_p]^*_{\text{Ref}}$ represents any right generalized inverse.

The control law is given by Eq. (12):

$$u_p^* = \left[\frac{\partial F_p^{\text{Control}}}{\partial u_p}\right]^*_{\text{Ref}} \{f_1(x_p) + f_2(\dot{x}_{\text{CM}})\} \quad (42)$$

Error Dynamics

It now remains to determine a control model function that ensures stability of the errors. Recall that the control model function is absolutely arbitrary, so long as it vanishes identically when $e = \{0\}$. Thus it can be linear or nonlinear, and for implementation requires only that it be computable.

The error dynamics are defined according to:

$$e \equiv \begin{Bmatrix} e^1 \\ e^2 \\ e^3 \end{Bmatrix} = \begin{Bmatrix} x_m^1 - x_p^1 \\ x_m^2 - x_p^2 \\ x_m^3 - x_p^3 \end{Bmatrix}, \qquad e^1 \equiv \begin{Bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{Bmatrix}, \text{ etc.}$$

and are given by [compare with Eq. (22)]:

$$\begin{aligned} \dot{e}_1 &= ½(-q_1P - q_2Q - q_3R)_m - ½(-q_1P - q_2Q - q_3R)_p \\ \dot{e}_2 &= ½(q_0P - q_3Q + q_2R)_m - ½(q_0P - q_3Q + q_2R)_p \\ \dot{e}_3 &= ½(q_3P + q_0Q - q_1R)_m - ½(q_3P + q_0Q - q_1R)_p \\ \dot{e}_4 &= ½(-q_2P + q_1Q + q_0R)_m - ½(-q_2P + q_1Q + q_0R)_p \end{aligned} \quad (43)$$

$$\begin{aligned} \dot{e}_5 &= [T_m^{32}g - R_mU_m + P_mW_m + (Y/m)(x_m, \dot{x}_m, \eta)] \\ &\quad - [T_p^{32}g - R_pU_p + P_pW_p + (Y/m)(x_p, \dot{x}_p, \eta)] \\ \dot{e}_6 &= [T_m^{33}g - P_mV_m + Q_mU_m + (Z/m)(x_m, \dot{x}_m, \eta)] \\ &\quad - [T_p^{33}g - P_pV_p + Q_pU_p + (Z/m)(x_p, \dot{x}_p, \eta)] \end{aligned} \quad (44)$$

$$\begin{aligned} \dot{e}_7 &= -f_{\text{CM}}^7(x_m, x_p), & \dot{e}_8 &= -f_{\text{CM}}^8(x_m, x_p) \\ \dot{e}_9 &= -f_{\text{CM}}^9(x_m, x_p), & \dot{e}_{10} &= -f_{\text{CM}}^{10}(x_m, x_p) \end{aligned} \quad (45)$$

Equations (43–45) may be linearized about the condition of zero error, and the control model functions become linear feedback gains. That is, for linearized error dynamics.

$$f_{\text{CM}}(x_m, x_p) = K_e \Delta e \quad (46)$$

where K_e is a feedback gain matrix (here, of dimension 4 × 10) operating on perturbations in the error Δe. The k_{ij} in K_e would have to be determined at various reference flight conditions, and stored for later use during flight. Alternatively, the nonlinear form of Eq. (43–45) may be treated directly by the suitable choice of a nonlinear control model.

Nonlinear Control Model

The development of nonlinear control model functions is based on the consideration of a candidate Lyapunov function $V = e^T\Gamma e$, with $\Gamma \equiv \text{diag}(\gamma_1 \ldots \gamma_{10})$, $\Gamma > 0$ (compare with Ref. 8, pp. 77–81). We then require control model functions such that

$$½\dot{V} = e^T\Gamma\dot{e} \le 0 \;\forall\; e, \qquad e^T\Gamma\dot{e} = 0 \Leftrightarrow e = \{0\} \quad (47)$$

With the notion that the orientation of the airplane (Euler parameters) should be controllable through the three moment equations, we first examine the contributions to Eq. (47) of the seven associated errors. We use the notation x_{m1} and x_{p1}, etc., to denote the model and plant states, respectively:

$$\begin{aligned} 2e_1\gamma_1\dot{e}_1 &= \gamma_1[x_{m1}(-x_{m2}x_{m7} - x_{m3}x_{m8} - x_{m4}x_{m9}) \\ &\quad - x_{m1}(-x_{p2}x_{p7} - x_{p3}x_{p8} - x_{p4}x_{p9}) \\ &\quad - x_{p1}(-x_{m2}x_{m7} - x_{m3}x_{m8} - x_{m4}x_{m9}) \\ &\quad + x_{p1}(-x_{p2}x_{p7} - x_{p3}x_{p8} - x_{p4}x_{p9})] \end{aligned} \quad (48)$$

with similar expressions for $2e_2\gamma_2\dot{e}_2 \ldots 2e_4\gamma_4\dot{e}_4$. Examination of these equations shows that half the terms cancel if we set

$$\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 \equiv \gamma_q \quad (49)$$

Following this, a bit of algebra yields:

$$\begin{aligned} \sum_{i=1}^{4} 2e_i\gamma_i\dot{e}_i &= \gamma_q(-x_{m1}x_{p2} - x_{m4}x_{p3} + x_{m2}x_{p1} + x_{m3}x_{p4})e_7 \\ &\quad + \gamma_q(-x_{m1}x_{p3} - x_{m2}x_{p4} + x_{m3}x_{p1} + x_{m4}x_{p2})e_8 \\ &\quad + \gamma_q(-x_{m1}x_{p4} - x_{m3}x_{p2} + x_{m2}x_{p3} + x_{m4}x_{p1})e_9 \end{aligned} \quad (50)$$

With respect to the errors in P, Q, and R:

$$\sum_{i=7}^{9} e_i\gamma_i\dot{e}_i = -\sum_{i=7}^{9} e_i\gamma_i f_{\text{CM}}^i = -\gamma_7 f_{\text{CM}}^7 e_7 - \gamma_8 f_{\text{CM}}^8 e_8 - \gamma_9 f_{\text{CM}}^9 e_9 \quad (51)$$

From Eq. (50) and (51) we see that the three control model functions associated with P, Q, and R may be selected as:

$$\begin{aligned} f_{\text{CM}}^7 &= -\lambda_7 e_7 \\ &\quad + (\gamma_q/2\gamma_7)(-x_{m1}x_{p2} - x_{m4}x_{p3} + x_{m2}x_{p1} + x_{m3}x_{p4}) \\ f_{\text{CM}}^8 &= -\lambda_8 e_8 \\ &\quad + (\gamma_q/2\gamma_8)(-x_{m1}x_{p3} - x_{m2}x_{p4} + x_{m3}x_{p1} + x_{m4}x_{p2} \\ f_{\text{CM}}^9 &= -\lambda_9 e_9 \\ &\quad + (\gamma_q/2\gamma_9)(-x_{m1}x_{p4} - x_{m3}x_{p2} + x_{m2}x_{p3} + x_{m4}x_{p1}) \end{aligned} \quad (52)$$

The result of this selection is:

$$\sum_{i=1}^{4} e_i\gamma_i\dot{e}_i + \sum_{i=7}^{9} e_i\gamma_i\dot{e}_i = \sum_{i=7}^{9} \lambda_i e_i^2 \quad (53)$$

This means that the portion of $\dot{V}$ arising from errors in the Euler parameters and the body angle rates may be made negative by the appropriate selection of λ_1, λ_2, and λ_3, independent of errors in the linear velocities. The parameters $\gamma_{7..9}$ are redundant in Eq. (52), and may be set to 1. γ_q becomes an error weighting parameter for all of the Euler parameters.

We proceed by assuming that the errors in angles (Euler parameters) and body angle rates are negligible. This is justified by Eq. (53) and the subsequent discussion. Applying this assumption to Eq. (44) and replacing the states with their generic (x_i) notation results in

$$\begin{aligned} \dot{e}_5 &= -x_9 e_{10} + x_7 e_6 \\ &\quad + (Y/m)(\boldsymbol{x}_m, \dot{\boldsymbol{x}}_m, \eta) - (Y/m)(\boldsymbol{x}_p, \dot{\boldsymbol{x}}_p, \eta) \\ \dot{e}_6 &= -x_7 e_5 + x_8 e_{10} \\ &\quad + (Z/m)(\boldsymbol{x}_m, \dot{\boldsymbol{x}}_m, \eta) - (Z/m)(\boldsymbol{x}_p, \dot{\boldsymbol{x}}_p, \eta) \end{aligned} \tag{54}$$

In Eq. (54), x_7, x_8, and x_9 (P, Q, and R) are the nominal (zero error) values of these states. We now expand the forces in the usual manner, with $Y_v \equiv [\partial Y/\partial V]_{\text{Ref}}$, etc., as:

$$\begin{aligned} Y(\boldsymbol{x}_m, \dot{\boldsymbol{x}}_m, \eta) &= Y_v V + Y_p P + Y_r R + Y_\eta \delta\eta \\ &= Y_v x_5 + Y_p x_7 + Y_r x_9 + Y_\eta \delta\eta \\ Z(\boldsymbol{x}_m, \dot{\boldsymbol{x}}_m, \eta) &= Z_u U + Z_w W + Z_q Q + Z_{\dot{w}} \dot{W} + Z_\eta \delta\eta \\ &= Z_u x_{10} + Z_w x_6 + Z_q x_8 + Z_{\dot{w}} \dot{x}_6 + Z_\eta \delta\eta \end{aligned} \tag{55}$$

We take errors in body axis rates to be zero, note that we have required the stability derivatives of this type equation to be the same for plant and model, and assume that

$$(1 - Z_{\dot{w}}/m) \approx 1$$

Substituting Eq. (55) into Eq. (54), we have

$$\begin{aligned} \dot{e}_5 &= (Y_v/m)e_5 + x_7 e_6 - x_9 e_{10} \\ \dot{e}_6 &= -x_7 e_5 + (Z_w/m)e_6 + [x_8 + (Z_u/m)]e_{10} \end{aligned} \tag{56}$$

Now take $\gamma_5 = \gamma_6 \equiv \gamma_V$, add the remaining contributions to V, with the result:

$$\begin{aligned} \sum_{i=5,6,10} e_i \gamma_i \dot{e}_i &= \gamma_V \left(\frac{Y_v}{m} e_5^2 + \frac{Z_w}{m} e_6^2 \right) \\ &\quad + \gamma_V \left(-x_9 e_5 + x_8 e_6 + \frac{Z_u}{m} e_6 - \frac{m\gamma_{10}}{\gamma_V} f_{cm}^{10} \right) e_{10} \end{aligned} \tag{57}$$

The terms Y_v and Z_w are rate damping terms and are negative. The last term in Eq. (57) is dealt with by taking:

$$f_{\text{CM}}^{10} = (\gamma_V/f_{10})[-x_9 e_5 + x_8 e_6 + (Z_u/m)e_6] - \lambda_{10} e_{10} \tag{58}$$

The parameter γ_{10} plays no role in this formulation, and may be set to $\gamma_{10} = 1$. As a result, we have:

$$\sum_{i=5,6,10} e_i \gamma_i \dot{e}_i = \frac{\gamma_V}{m}(Y_v e_5^2 + Z_w e_6^2) + \lambda_{10} e_{10}^2 \tag{59}$$

In short, the control model functions are defined by Eq. (52) and (58). This satisfies Eq. (47), which is the sum of Eq. (53) and (59). These control model functions are added to the appropriate model state rates [Eq. (13)] to yield the error-correcting control model state rates for use in the control law, Eq. (42).

Model Trajectory Generation

The control law requires the time histories of the model's body-axis states and of certain of its state rates. These are normally obtained by flying the model in real time, and continuously feeding the integration results to the control law for computation. Here, however, we have elected to specify a trajectory or maneuver for the airplane to execute. The model is therefore taken to have the same physical properties (mass, inertia, etc.), and the same aerodynamic force characteristics, as the airplane. This makes the problem one of dynamic inversion, and satisfies the requirements placed on the types 1 and 2 equations of motion. The problem is to determine the body-axis angular rates and accelerations required to perform the maneuver. These become inputs to the control law wherein, loosely, the moments (and hence controls) required to generate them are determined. The control law further assures the nominal stability of the errors between model and airplane, and continuously corrects for deviations of the airplane from the desired flight path.

The maneuvers prescribed for the design challenge are described in the example that follows. The required model states and state rates were determined off line using algorithms developed by one of the authors (Munro). Complete descriptions of these algorithms may be found in Ref. 9.

Example

The airplane is represented by a nonlinear, six-degree-of-freedom aerodynamic and engine data base provided by NASA Ames-Dryden Flight Research Center. The model utilizes table look up and interpolation to yield force and moment coefficient information. (Because of possible confusion over two uses of the word "model," the aerodynamic model will be referred to as if it were an actual airplane throughout the rest of this paper.)

The maneuvers performed consisted of steady, symmetric level flight with an external disturbance (vertical gust), and of transition from steady level flight into a two-g turn. The reference conditions for the model are at 9800 ft and Mach 0.5. The physical characteristics of the model (mass, moments of inertia, etc.) are the same as those of the airplane.

Controls are $\boldsymbol{u}_p^T = \{\delta_{HTS}\ \delta_{HTD}\ \delta_{AIL}\ \delta_R\ \delta_T\}$, where the δ is taken to mean the deflection measured from reference (zero) displacement. These controls are defined as follows: δ_{HTS}, horizontal tail, symmetric; δ_{HTD}, horizontal tail, differential; δ_{AIL}, aileron; δ_R, rudder; and δ_T, thrust.

Simulation

The model states and state rates were provided to the simulation as functions of time, in 0.02 s increments, in data files which were read as required by the control law. The control law is as described above. Values for the λ_i of the control model functions $f_{\text{CM}}(x_m, x_p)$, and the γ_q and γ_V that appears in the Lyapunov function, were established by systematically varying the values of these parameters while observing the error and control activity during simulations of the maneuvers. Error feedback gains that give qualitatively satisfactory responses with minimal control saturation are as follows:

Euler parameter errors (γ_q)	1000.0
V, W errors (γ_V)	30.0
Roll rate error (λ_7)	−20.0
Pitch rate error (λ_8)	−20.0
Yaw rate error (λ_9)	−20.0
X-Velocity error (λ_{10})	−20.0

In this problem, there are five airplane controllers. The control matrix in Eq. (39) was found to be of full rank throughout the flight regimes examined, and the minimum norm solution was used in formulating the control law. While the minimum norm solution did yield some control saturation (possibly avoidable through the use of a different generalized inverse), the effect of this saturation is not catastrophic.

The control law, Eq. (42), requires knowledge or estimates of several airplane parameters. The best estimates of these forces and moments are usually obtained from the airplane stability derivatives. Thus calculation of the control law requires all the airplane stability and control derivatives. Because these vary with flight condition, primarily Mach and angle of attack, some updating, or scheduling, of these parameters may be required for satisfactory performance.

The simulation used below had provisions for updating all of the airplane stability and control derivatives used in the calculation of the control law, including the precomputed generalized inverses. These parameters were stored as third-order polynomial fits as functions of angle of attack only. The frequency of updating the parameters was selected by specifying a discrete step size in angle of attack. As the airplane angle of attack reached the next 10-deg increment of angle of attack, the parameters were re-evaluated.

Results and Discussion

Figures 1 through 12 show the results of the simulations. In these figures, Euler parameters have been converted to Euler angles, and linear velocity components have been converted to total velocity, angle of attack, and sideslip angle.

Figures 1 and 2 show the results obtained for steady level flight with a vertical gust input between 5–9 s. Model following errors were unremarkable in all states except total velocity and angle of attack. The addition of an external component of velocity causes an error to be generated for the duration of the gust. As the gust is removed, the control law drives the airplane states back to those of the model. As seen in Fig. 3, the throttle was saturated by the attempt to control the velocity. For this symmetric maneuver, control saturation is not catastrophic, and

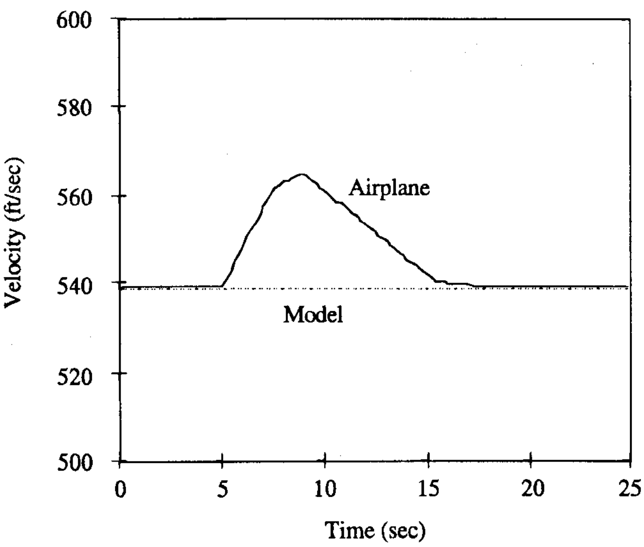

Fig. 1 Velocity response, steady level flight with vertical gust.

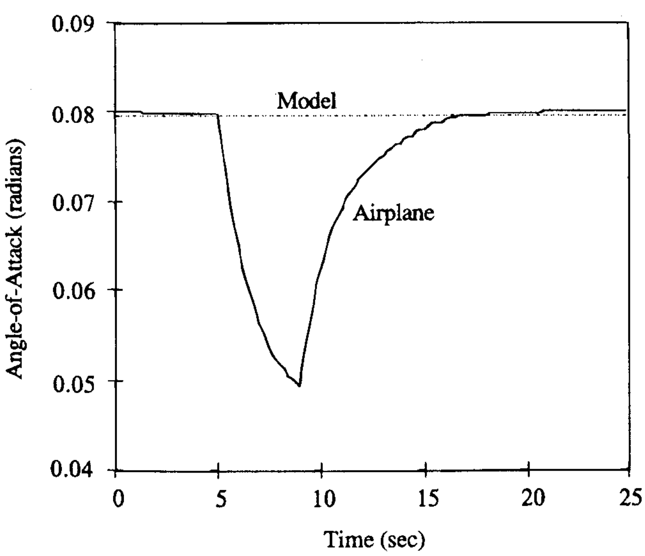

Fig. 2 Angle-of-attack response, steady level flight with vertical gust.

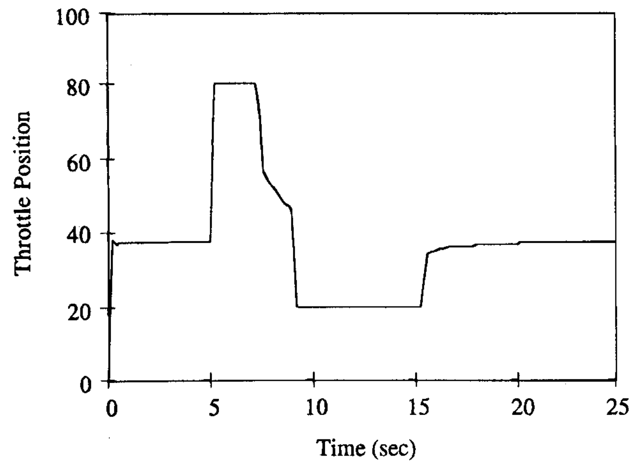

Fig. 3 Throttle required, steady level flight with vertical gust.

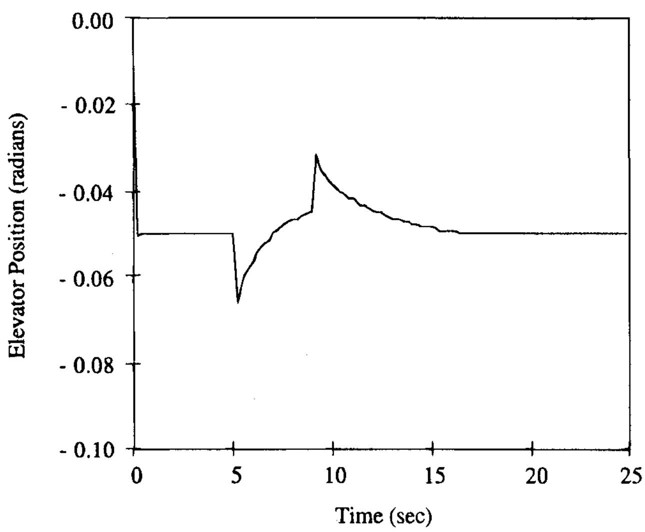

Fig. 4 Elevator required, steady level flight with vertical gust.

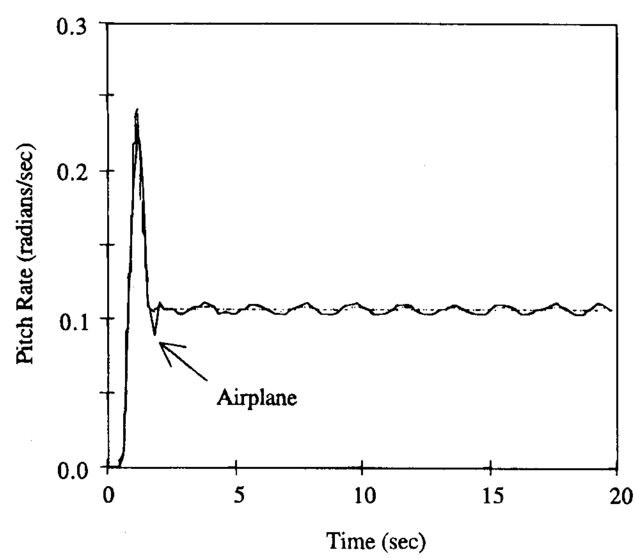

Fig. 5 Pitch rate response, 2-*g* level turn.

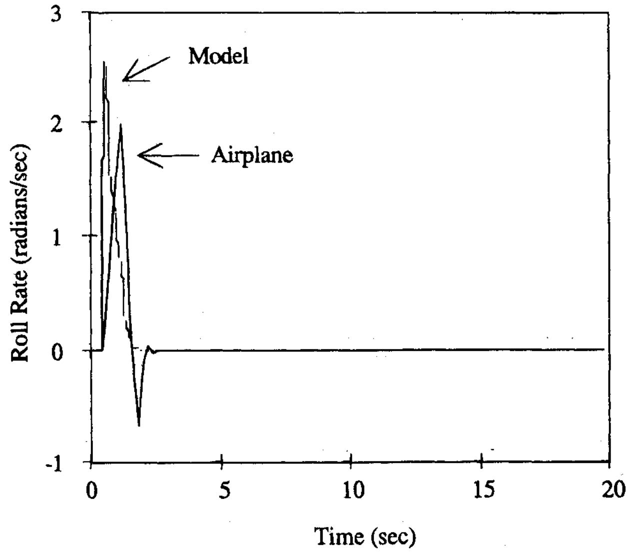

Fig. 6 Roll rate response, 2-*g* level turn.

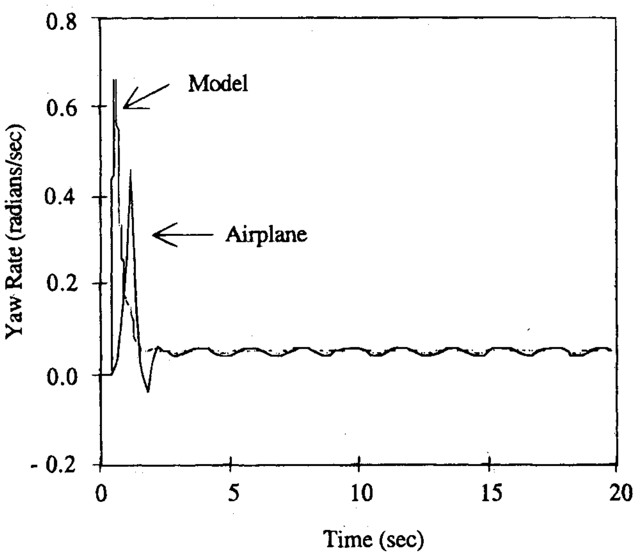

Fig. 7 Yaw rate response, 2-*g* level turn.

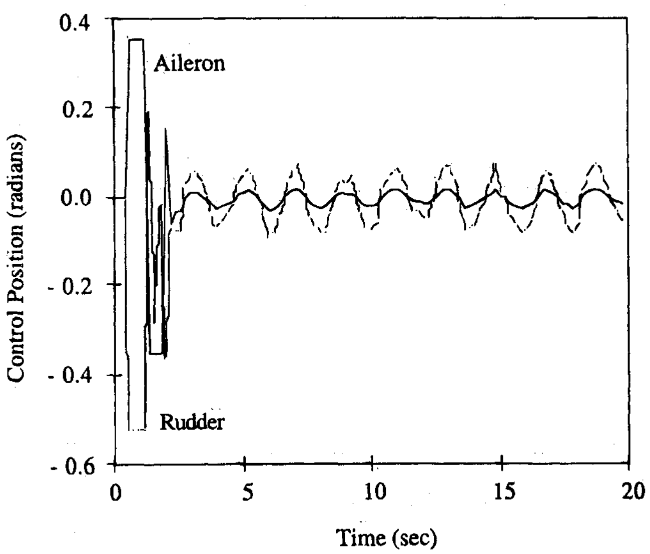

Fig. 8 Aileron and rudder required, 2-*g* level turn.

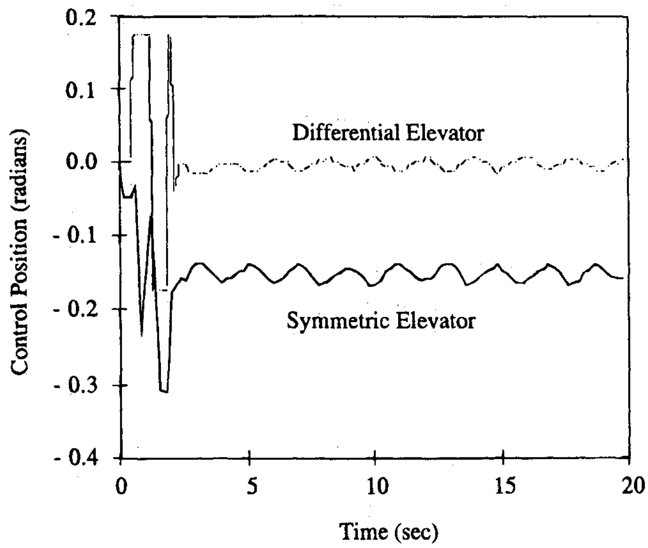

Fig. 9 Elevator required, 2-*g* level turn.

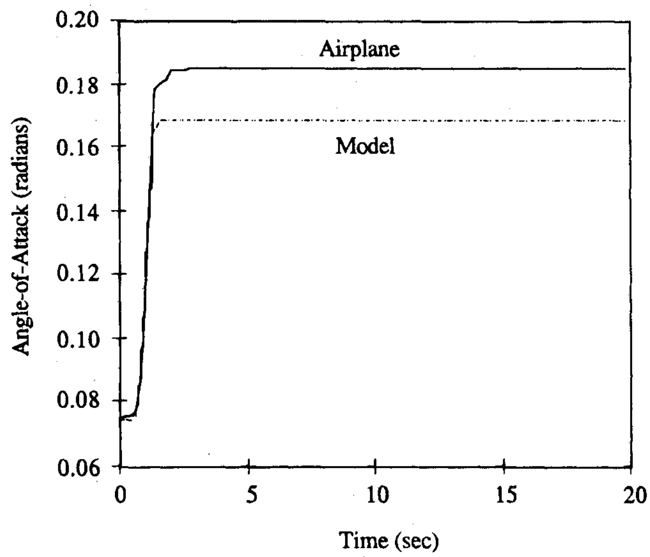

Fig. 10 Angle-of-attack response, 2-*g* level turn.

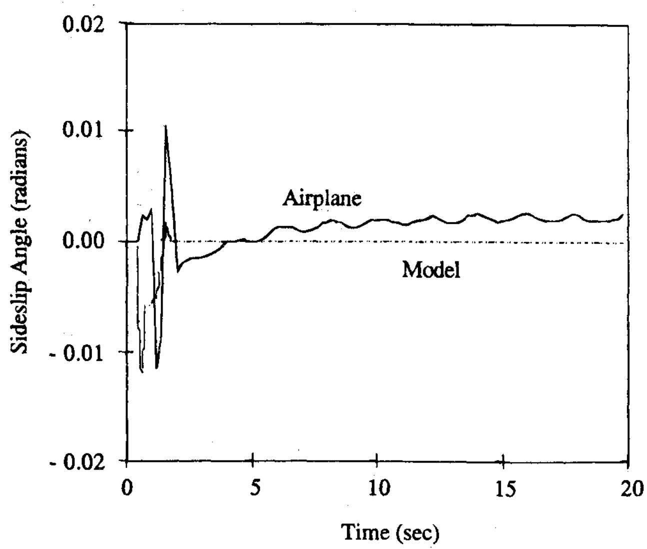

Fig. 11 Sideslip response, 2-*g* level turn.

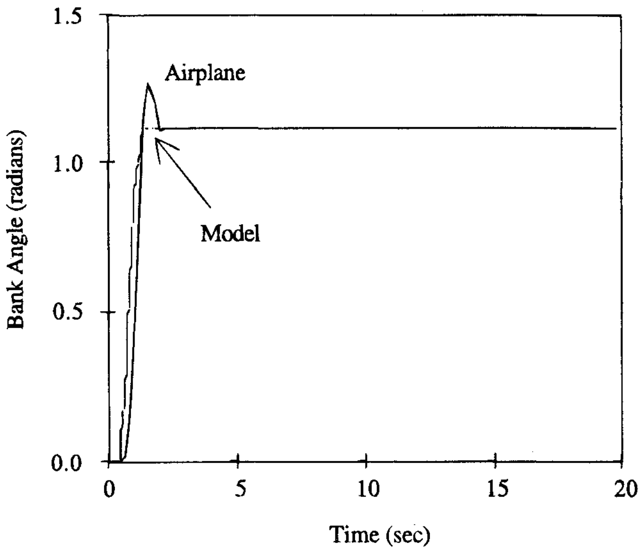

Fig. 12 Bank-angle response, 2-*g* level turn.

serves only to delay the error correction. The elevator remained within limits throughout (Fig. 4), reflecting the fact that it is primarily a pitch rate control, and that there were no pitch rate requirements in this maneuver.

The remaining figures are the result of the 2-*g* turn simulation. The angle rate results, Figs. 5–7, show the transition from straight to turning flight after one second. The airplane follows the pitch rate well (Fig. 5), but lags the model in roll and yaw rate (Figs. 6 and 7). This lag is a result of lateral control saturation, as seen in Figs. 8 and 9. The small oscillations in model pitch and yaw rates seen following the transition could not be explained, but are believed to be due to numerical truncation errors in the algorithm used in obtaining state derivatives.

Figures 10 and 11 show the time histories of angle of attack and sideslip. The steady-state error in angle of attack is attributed to the simplifying assumption made with regard to the model's aerodynamic lift characteristics, wherein only angle of attack dependency was assumed. Likewise, the model's side-force characteristics were taken to be dependent on sideslip only, and presumably caused the sideslip errors seen in Fig. 11.

Finally, Fig. 12 shows the bank-angle response of the airplane. The slight overshoot as the final bank angle is reached is again due to control saturation, as seen in Fig. 8.

Conclusions

The notion of a standard form for nonlinear model-following control law development gives rise to a nonlinear form of the control model used in its formulation. Consideration of a simple Lyapunov function for the error allows selection of a control model function that assures stability of the error for nominal cases. The error feedback gains incorporated into the control model functions caused the airplane in this simulation to faithfully follow the trajectory of the model, so long as the controllers are not saturated. During periods of control saturation, the model-following control law is no longer valid, and system errors grow. The removal of the excess requirements for control deflections re-establishes the validity of the control law, and stable error dynamics are demonstrated.

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